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A METHOD FOR CONTINUING FAMILIES OF PERIODIC SOLUTIONS OF LAGRANGIAN SYSTEMS*

S.R. KARIMOV and A.G. SOKOL'SKII

An autonomous multidimensional Lagrangian system of differential equations depending on parameters is considered. A predictor-corrector method is proposed for constructing a family of periodic solutions (including retrograde solutions) obtained from a given solution by variation of the parameters.

In the context of the wide range of problems in classical and celestial mechanics that are described by Lagrangian systems of differential equations, it is particularly interesting to study non-isolated **periodic** solutions of such equations, parametrized by both extrinsic and intrinsic parameters (the latter are represented by the initial conditions of the solution, such as the energy constant). A family parametrized by the energy constant is known as a natural family /1, 2/.

The classical example of a natural family of periodic orbits is provided by the Lyapunov periodic motions /3/ **originating from the** equilibrium position of a Hamiltonian system. Existing methods of investigating them /4, 5/ are based on the introduction **of local coordinates** in the neighbourhood of a periodic solution and subsequent normalization of the equations of the perturbed motion, and the construction of the solution as a series with respect to a **small parameter**, where the latter characterizes the deviation of the motion from the equilibrium position.

In a more complicated situation the generating solution is known only in terms of its initial conditions and period, the solution itself being obtained by numerical integration of the original system. Consequently, any method for continuing a (not necessarily Lyapunov) family by expansion with respect to the parameters is necessarily numerical.

On the available variety of such methods we mention a series of papers by Sarychev and Sazonov (for references see /6/), who have worked out a highly efficient method for solving the boundary-value problems that arise in this context. This paper draws on ideas similar to those of /7-9/**, (**See also Sokol'skii A.G. and Khovanskii S.A., A computation algorithm for continuing periodic solutions of two-dimensional Hamiltonian systems as functions of the parameters, Moscow, MAI, 1986. Deposited at VINITI, 4.06.86, 4042-86.) where a predictor-corrector method for continuation of periodic solutions is worked out for systems with two degrees of freedom. This method will be generalized here to multidimensional systems, **for which Birkhoff's** theorem about the reduction of a two-dimensional system to canonical form is no longer valid; it is nevertheless possible to reduce the boundary-value problem to a Cauchy problem by a special choice of local coordinates.

1. Formulation of the problem. Consider a generalized conservative mechanical system with $J+1$ degrees of freedom, depending on $K-1$ parameters. Its Lagrangian has the form

$$L^* = \frac{1}{2}\sum_{j,\,i=1}^{J+1} l_{ji}(\mathbf{q},\mathbf{p})\, q_j^{\cdot} q_i^{\cdot} + \sum_{j=1}^{J+1} l_j(\mathbf{q},\mathbf{p})\, q_j^{\cdot} + l_0(\mathbf{q},\mathbf{p})$$

$$\mathbf{q} = (q_1,\ldots,q_{J+1})^T,\quad \mathbf{q}^{\cdot} = (q_1^{\cdot},\ldots,q_{J+1}^{\cdot})^T,\quad \mathbf{p} = (p_1,\ldots,p_{K-1})^T$$

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Here $\mathbf{q}, \mathbf{q}^{\cdot}$ are the generalized coordinates and velocities, $\mathbf{p}$ are the parameters of the system, and l_{ji}, l_j, l_0 are sufficiently smooth (in particular, analytic) functions of their variables, such that the matrix $L = \{l_{ji}\}$ is symmetric and positive-definite.

The corresponding system of equations admits of a Jacobi-type energy integral: $C = \frac{1}{2} q^{\cdot T} L q^{\cdot} - l_0$, where $C = h$ is the energy constant, which depends on the initial conditions. If the energy constant is fixed (i.e., the only motions considered are those confined to the hyperplane $h = \text{const}$), we may treat h as an additional parameter of the mechanical system, all other motions of which are derived (parametrized) from these motions by varying h. Thus, we put $p_K = h$, so that $\mathbf{p}$ is now a K-dimensional vector; in addition, we put $\mathbf{l} = (l_1, \ldots, l_{J+1})^T$, $W(\mathbf{q}, \mathbf{p}) = l_0 + h$ (this will enable us later to confine our attention to cases in which the energy integral is identically zero).

In the notation just described, the Lagrangian and energy integral are written as follows:

$$L^* = \frac{1}{2} \mathbf{q}^{\cdot T} L \mathbf{q}^{\cdot} + \mathbf{l}^T \mathbf{q}^{\cdot} + W \tag{1.1}$$

$$C = \frac{1}{2} \mathbf{q}^{\cdot T} L \mathbf{q}^{\cdot} - W \equiv 0 \tag{1.2}$$

The equations of motion are

$$\mathbf{q}^{\cdot\cdot} = \mathbf{f}(\mathbf{q}, \mathbf{q}^{\cdot}, \mathbf{p}), \quad \mathbf{f} = L^{-1} \boldsymbol{\varphi}, \quad \boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_{J+1})^T \tag{1.3}$$

$$\varphi_j = \sum_{i, \alpha=1}^{J+1} \left({}^1\!/_2 \frac{\partial l_{i\alpha}}{\partial q_j} - \frac{\partial l_{ji}}{\partial q_\alpha} \right) q_i^{\cdot} q_\alpha^{\cdot} + \sum_{i=1}^{J+1} \left(\frac{\partial l_i}{\partial q_j} - \frac{\partial l_j}{\partial q_i} \right) q_i^{\cdot} + \frac{\partial W}{\partial q_j}$$

Suppose that for some fixed parameter vector $\mathbf{p} = \mathbf{P}$ the initial values are known for some solution of Eq.(1.3), say

$$\mathbf{q} = \mathbf{Q}(t, \mathbf{P}), \quad \mathbf{q}^{\cdot} = \mathbf{Q}^{\cdot}(t, \mathbf{P}) \tag{1.4}$$

where the solution has a period $T = T(\mathbf{P})$, i.e.,

$$\mathbf{Q}(0, \mathbf{P}) = \mathbf{Q}(T, \mathbf{P}), \quad \mathbf{Q}^{\cdot}(0, \mathbf{P}) = \mathbf{Q}^{\cdot}(T, \mathbf{P}) \tag{1.5}$$

In this situation the functions (1.4) themselves may be determined by numerical integration of Eqs.(1.3) in the interval $t \in [0, T]$.

The problem we consider is to construct and investigate periodic solutions which are analytic continuations (with respect to the parameters) of (1.4), i.e., we wish to determine solutions

$$\mathbf{q} = \mathbf{q}(t, \mathbf{p}), \quad \mathbf{q}^{\cdot} = \mathbf{q}^{\cdot}(t, \mathbf{p}) \tag{1.6}$$

such that

$$\lim \mathbf{q}(t, \mathbf{p}) = \mathbf{Q}(t, \mathbf{P}), \quad \lim \mathbf{q}^{\cdot}(t, \mathbf{p}) = \mathbf{Q}^{\cdot}(t, \mathbf{P}),$$

$$\lim T(\mathbf{p}) = T(\mathbf{P}) \quad (\mathbf{p} \to \mathbf{P}) \tag{1.7}$$

$$\mathbf{q}(0, \mathbf{p}) = \mathbf{q}(T(\mathbf{p}), \mathbf{p}), \quad \mathbf{q}^{\cdot}(0, \mathbf{p}) = \mathbf{q}^{\cdot}(T(\mathbf{p}), \mathbf{p}) \tag{1.8}$$

Eqs.(1.8) are the periodicity conditions. Conditions (1.7) state that the solution (1.6) belongs to the family of periodic solutions generated by the solution (1.4). If there are no solutions (1.6) satisfying conditions (1.7) and (1.8), we shall say that the family is "dead" /1/; if there is more than one solution satisfying the conditions, we shall say that the family branches.

This formulation of the problem is essentially the same as the classical one. The most popular method for solving it is the small-parameter method, where the question of existence and branching is adequately answered by Poincaré's Theorem and its extensions. The conditions derived below for the existence of a family are analogous to Poincaré's conditions.

Note that it will be sufficient to determine not the solutions (1.6) themselves, but only initial conditions $\mathbf{q}(0, \mathbf{p})$, $\mathbf{q}^{\cdot}(0, \mathbf{p})$ satisfying the periodicity conditions (1.8) and continuity conditions (1.7) at $t \equiv 0$. In addition, it should be noted that the numerical approach to the solution of our problem obliges us to replace the infinitesimal increments of the parameters under these conditions by small but finite increments. If this is indeed done and computations show that the family is "dead" (or branches), one must then take smaller increments and repeat the computation. Whether the goal can be attained in a finite number of steps depends on the capacity of the available computers.

2. Introduction of local coordinates. Let (1.4) be a known solution of Eqs.(1.3) with integral (1.2) and parameter values $\mathbf{P}$. Let (1.6) be another solution of Eqs.(1.3), corresponding to parameter values $\mathbf{p}$. We put

$$\boldsymbol{\pi} = \mathbf{p} - \mathbf{P}, \quad \boldsymbol{\xi} = \mathbf{q} - \mathbf{Q} \tag{2.1}$$

We shall use this setup in three versions: in stability analysis, in the predictor part of the method and in its corrector part. The equations for the local coordinates ξ (and for the relevant transformation of the equations) will be analogous in all three cases. We therefore proceed at once to a uniform presentation of the formal arguments and transformation of the equations.

For the moment, let us assume that the increments π and ξ are independent and small to the same order of magnitude. Retaining only first-order terms in the expansions, we obtain the following linear equations for ξ:

$$\xi^{\cdot\cdot} = f_Q\xi + f_{Q^\cdot}\xi^\cdot + f_P\pi \tag{2.2}$$
$$f_Q = \partial\mathbf{f}/\partial\mathbf{q}|_0, \quad f_{Q^\cdot} = \partial\mathbf{f}/\partial\mathbf{q}^\cdot|_0, \quad f_P = \partial\mathbf{f}/\partial\mathbf{p}|_0$$

where the zero subscript means that after differentiation we substitute $\mathbf{q} = \mathbf{Q}$, $\mathbf{p} = \mathbf{P}$, i.e., $\xi = \mathbf{0}$, $\pi = \mathbf{0}$.

Eqs.(2.2) admit of the following integral, obtained from (1.2) by retaining first-order terms:

$$w = g_Q\xi + g_{Q^\cdot}\xi^\cdot + g_P\pi \equiv 0 \tag{2.3}$$
$$g_Q = \frac{1}{2}\mathbf{Q}^{\cdot T}L_Q\mathbf{Q}^\cdot - W_Q, \quad g_{Q^\cdot} = \mathbf{Q}^{\cdot T}L, \quad g_Q = \frac{1}{2}\mathbf{Q}^{\cdot T}L_P\mathbf{Q}^\cdot - W_P$$
$$L_Q = \partial L/\partial\mathbf{q}|_0, \quad W_Q = \partial W/\partial\mathbf{q}|_0, \quad L_P = \partial L/\partial\mathbf{p}|_0,$$
$$W_P = \partial W/\partial\mathbf{p}|_0$$

Put

$$V(t) = |\mathbf{Q}^\cdot(t)| = \Big[\sum_{j=1}^{J+1} Q_j^{\cdot 2}(t)\Big]^{1/2} \tag{2.4}$$

i.e., V is the absolute value of the instantaneous velocity along an orbit.

We shall assume that (1.4) is not an equilibrium position (otherwise one can apply the methods of /4, 5/). Consequently, $V(t) \not\equiv 0$. We shall also assume that throughout the orbit $V(t) \neq 0$.

Under these assumptions, there is a tangent at very point of the orbit in configuration space. We may therefore attach a moving system of coordinates to the orbit: one of the axes is directed along the velocity vector $\mathbf{Q}^\cdot$ and the others lie in the plane normal to the orbit.

Let S be the transformation matrix to the new coordinate system; let its last column $\mathbf{s}(t) = \mathbf{Q}^\cdot(t)/V(t)$ be the unit vector tangent to the orbit, while the first J columns $\mathbf{s}_j$ ($j = 1, \ldots, J$) lie in the plane normal to the orbit, i.e., they are orthogonal to the vector $\mathbf{s}$. The orbit $\mathbf{Q}(t)$ in the configuration space $\{\mathbf{Q}\}$, its tangent and its normal (hyper)plane at time t and the new basis vectors $\mathbf{s}_1, \ldots, \mathbf{s}_J$, $\mathbf{s}$ are shown in Fig.1.

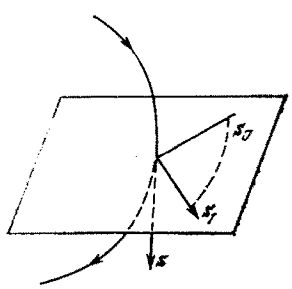

Fig.1

Thus,

$$S = \{R, \mathbf{s}\}, \quad R = \{\mathbf{s}_1, \ldots, \mathbf{s}_J\}, \quad \dim R = (J+1)\times J \tag{2.5}$$
$$\mathbf{s}^T\mathbf{s} = 1, \quad R^T\mathbf{s} = \mathbf{0}, \quad S^{-1}\mathbf{s} = \mathbf{e} = (0, \ldots, 0, 1)^T$$

In addition, let R^* denote the matrix S^{-1} without its last column $\mathbf{s}^*$, i.e., by (2.5), $R^*\mathbf{s} = \mathbf{0}$, $\mathbf{s}^*\mathbf{s} = 1$. Incidentally, in practice it is convenient to let S be an orthogonal matrix, but this will not be necessary here. Clearly, if (1.4) is a periodic solution then the matrix S will also be periodic.

Let $\mathbf{x}$ denote the vector of local coordinates in the new coordinate system; we express it as

$$\mathbf{x} = \begin{Bmatrix}\mathbf{n}\\ m\end{Bmatrix},$$
$$\dim\mathbf{n} = J\times 1, \quad \dim m = 1\times 1$$

In geometrical terms, this means that m is a displacement (or perturbation) along the orbit and the components of $\mathbf{n}$ are displacements (perturbations) along the normals to the orbit. The relationship between the old and new local coordinates is given by the formulae

$$\xi = S\mathbf{x} = R\mathbf{n} + \mathbf{s}m; \quad \mathbf{x} = S^{-1}\xi; \quad \mathbf{n} = R^*\xi, \quad m = \mathbf{s}^*\xi \tag{2.6}$$
$$\xi^\cdot = S^\cdot\mathbf{x} + S\mathbf{x}^\cdot = R^\cdot\mathbf{n} + R\mathbf{n}^\cdot + \mathbf{s}^\cdot m + \mathbf{s}m^\cdot \tag{2.7}$$

Substituting these expressions into the integral (2.3), we obtain

$$w = g_n \mathbf{n} + g_{n'} \mathbf{n}' + g_m m + g_{m'} m' + g_P \pi \equiv 0 \qquad (2.8)$$
$$g_n = g_Q R + g_{Q'} R', \quad g_{n'} = g_{Q'} R$$
$$g_m = g_Q \mathbf{s} + g_{Q'} \mathbf{s}', \quad g_{m'} = g_{Q'} \mathbf{s}$$

Using formulae (2.3) and the relations

$$\dot{C} = \mathbf{Q}''^T L \mathbf{Q}' + \frac{1}{2} \mathbf{Q}'^T (L_Q \mathbf{Q}') \mathbf{Q}' - W_Q \mathbf{Q}' =$$
$$V [V' \mathbf{s}^T L \mathbf{s} + g_Q \mathbf{s} + g_{Q'} \mathbf{s}'] \equiv 0, \quad \mathbf{s}^T L \mathbf{s} = 2W/V^2 \qquad (2.9)$$

we can write the integral (2.8) as

$$w = (m' V - m V') \cdot 2W/V^2 + g_n \mathbf{n} + g_{n'} \mathbf{n}' + g_P \pi \equiv 0 \qquad (2.10)$$

We will now set up the equations for the new local coordinates, using the fact that $S\mathbf{x}'' = \xi'' - 2S'\mathbf{x}' - S''\mathbf{x}$. Substituting expressions (2.2), (2.6) and (2.7) into this equation, we obtain

$$S\mathbf{x}'' = (f_Q R + f_{Q'} R' - R'') \mathbf{n} + (f_{Q'} R - 2R') \mathbf{n}' +$$
$$(f_Q \mathbf{s} + f_{Q'} \mathbf{s}' - \mathbf{s}'') m + (f_{Q'} \mathbf{s} - 2\mathbf{s}') m' + f_P \pi \qquad (2.11)$$

Noting that

$$\mathbf{Q}''' = \mathbf{f}' = f_Q \mathbf{Q}' + f_{Q'} \mathbf{Q}'' = (V f_Q + V' f_{Q'}) \mathbf{s} + V f_{Q'} \mathbf{s}' \qquad (2.12)$$

we conclude that the vector coefficient of m in (2.11) has the form $-(V'/V)(f_{Q'}\mathbf{s} - 2\mathbf{s}') + (V''/V)\mathbf{s}$ (compare with the coefficient of m'). Thus, after substituting the expression for $m'V - mV'$ from the integral (2.10) and the expressions (2.8) into (2.11), we obtain the following equations:

$$\mathbf{x}'' = S^{-1} [F_n n + F_{n'} \mathbf{n}' + f_\pi \pi + (V''/V) m \mathbf{s}] \qquad (2.13)$$
$$F_n = [f_Q - f_s g_Q] R + [f_{Q'} - f_s g_{Q'}] R' - R'',$$
$$f_s = (V/2W)(f_{Q'} \mathbf{s} - 2\mathbf{s}')$$
$$F_{n'} = [f_{Q'} - f_s g_{Q'}] R - 2R', \quad F_\pi = f_P - f_s g_P$$

Recall that $S^{-1}\mathbf{s} = \mathbf{e}$ (see (2.5)). It is therefore obvious that instead of system (2.13) we can consider the following two linear equations:

$$\mathbf{n}'' = R^* [F_n \mathbf{n} + F_{n'} \mathbf{n}' + F_\pi \pi] \qquad (2.14)$$
$$m'' = (V''/V) m + \mathbf{s}^* [F_n \mathbf{n} + F_{n'} \mathbf{n}' + F_\pi \pi] \qquad (2.15)$$

A remarkable property of Eqs.(2.14) and (2.15), in fact the motive for the transformation (2.6), is that the equations for the normal coordinates $\mathbf{n}$ are independent of the tangential coordinate m. This means that one can first determine the normal displacements (perturbations) $\mathbf{n}$ and then, using the resulting $\mathbf{n}$, the tangential displacement (perturbation) m.

3. Predictor. Thus, let (1.4) be a known T-periodic solution of Eqs.(1.3) with integral (1.2) and parameter values $\mathbf{P}$. Assume that for parameter values $\mathbf{p} = \mathbf{P} + \pi$, where π are given parameter increments, there exists a periodic solution (1.6) satisfying the continuity and periodicity conditions (1.7) and (1.8). Our aim is to find (1.6). The algorithm for continuation with respect to the parameters falls into two stages. First (predictor) one finds corrections, linear in the parameter increments, to the initial conditions and period; then (corrector) one brings into play the non-linear nature of the corrections, having constructed a convergent iterative procedure in this section we will describe the predictor part of the method.

Introduce displacements ξ (local coordinates) in accordance with formulae (2.1) and express the new period as $T^* = T + \tau$, where $T = T(\mathbf{P})$, $T^* = T(\mathbf{p})$. We shall treat ξ and τ as first-order quantities with respect to π. The displacements will then satisfy Eqs.(2.2) with integral (2.3). After introducing the normal and tangential displacements $\mathbf{n}$ and m in accordance with the formulae of Sect.2, we arrive at Eqs.(2.14), (2.10). All the coefficients of $\mathbf{n}$ and m in these equations are T-periodic.

Using the periodicity of the solutions (1.4) and (1.6) and retaining only first-order terms, we obtain the following boundary conditions:

$$\mathbf{n}(0) = \mathbf{n}(T), \quad \mathbf{n}'(0) = \mathbf{n}'(T) \qquad (3.1)$$
$$m(0) = m(T), \quad m'(0) = m'(T) + V'(0)\tau \qquad (3.2)$$

Express $\mathbf{n}, m, \tau$ as linear combinations of the varied parameters:

$$\mathbf{n} = \sum_{k=1}^{K} \mathbf{n}^{(k)}\pi_k, \quad m = \sum_{k=1}^{K} m^{(k)}\pi_k, \quad \tau = \sum_{k=1}^{K} \tau^{(k)}\pi_k \tag{3.3}$$

and substitute these expressions into Eqs.(2.14), (2.10) and the boundary conditions (3.1), (3.2). Using the independence of the parameter increments π_k, we obtain K sets of such relations.

We will first consider the boundary-value problem for the normal displacements:

$$\frac{d}{dt}\mathbf{v}^{(k)} = \begin{Vmatrix} 0 & E_J \\ R^*F_n & R^*F_{n^{\cdot}} \end{Vmatrix} \mathbf{v}^{(k)} + \begin{Vmatrix} 0 \\ R^*F_{\pi_k} \end{Vmatrix}, \quad \mathbf{v}^{(k)} = \begin{Vmatrix} \mathbf{n}^{(k)} \\ \mathbf{n}^{\cdot(k)} \end{Vmatrix} \tag{3.4}$$

$$\mathbf{v}^{(k)}(0) = \mathbf{v}^{(k)}(T) \tag{3.5}$$

where F_{π_k} is the k-th column of the matrix F_π in (2.13), i.e., the result of differentiating the corresponding functions with respect to the k-th parameter p_k only; E_J is the identity matrix of order J.

The boundary-value problem (3.4), (3.5) can be reduced to a Cauchy initial-value problem. Indeed, the general solution of the inhomogeneous Eqs.(3.4) can be written as

$$\mathbf{v}^{(k)}(t) = N(t)\,\mathbf{v}^{(k)}(0) + \mathbf{v}^{(\pi_k)}(t) \tag{3.6}$$

Here $N(t)$ is the matrix of fundamental solutions of the homogeneous system normalized by the condition $N(0) = E_{2J}$, $\mathbf{v}^{(k)}(0)$ are the initial conditions for $\mathbf{v}^{(k)}(t)$, $\mathbf{v}^{(\pi_k)}(t)$ is a particular solution of the inhomogeneous equations with zero initial values, i.e., $\mathbf{v}^{(\pi_k)}(0) = \mathbf{0}$. Substituting (3.6) into the boundary conditions (3.5), we have

$$\mathbf{v}^{(k)}(0) = -[N(T) - E_{2J}]^{-1}\mathbf{v}^{(\pi_k)}(T) \tag{3.7}$$

Thus, (3.6) with the vector (3.7) is a solution of the boundary-value problem (3.4), (3.5).

To find the variation τ in the period, we rewrite the coefficient of π_k obtained from Eq.(2.10) as follows:

$$m^{\cdot(k)} = \frac{V^{\cdot}}{V} m^{(k)} - \frac{V}{2W}[g_v \mathbf{v}^{(k)} + g_{P_k}] \tag{3.8}$$

where $g_v = \{g_n, g_{n^{\cdot}}\}$ and g_{P_k} is the k-th element of the matrix row g_P in (2.3). Substituting the solution (3.6), into (3.8), we obtain the general solution of Eq.(3.8) in the form

$$m^{(k)}(t) = (V(t)/V(0))\, m^{(k)}(0) + \mu(t)\,\mathbf{v}^{(k)}(0) + \mu^{(\pi_k)}(t) \tag{3.9}$$

where the row vectors $\mu(t)$ and $\mu^{(\pi_k)}(t)$ are solutions of the following Cauchy problems:

$$\dot{\mu} = (V^{\cdot}/V)\mu - (V/(2W))g_v N, \quad \mu(0) = 0$$

$$\mu^{\cdot(\pi_k)} = (V^{\cdot}/V)\,\mu^{(\pi_k)} - (V/2W)\,[g_v \mathbf{v}^{(\pi_k)} + g_{P_k}], \quad \mu^{(\pi_k)}(0) = 0$$

The initial displacement $m^{(k)}(0)$ may be equated to zero, since displacement along the orbit does not affect it. Then, from the first periodicity condition (3.2) we obtain $m^{(k)}(0) = m^{(k)}(T) + V(0)\tau$, and consequently,

$$\tau^{(k)} = -m^{(k)}(T)/V(0) = [\mu(T)\,\mathbf{v}^{(k)}(0) + \mu^{(\pi_k)}(T)]/V(0) \tag{3.10}$$

Direct differentiation of solution (3.9) shows that the second periodicity condition (3.2) is automatically satisfied if $\tau^{(k)}$ is chosen in accordance with (3.10), and moreover

$$m^{(k)}(0) = 0, \quad m^{\cdot(k)}(0) = (-V(0)/(2W(0)))\,[g_v(0)\,\mathbf{v}^{(k)}(0) + g_{P_k}(0)] \tag{3.11}$$

Thus, to determine the new periodic motion one must integrate the following system of equations from $t = 0$ to $t = T$:

$$\mathbf{Q}^{\cdot\cdot} = f(\mathbf{Q}, \mathbf{Q}^{\cdot}, \mathbf{P}) \tag{3.12}$$

$$\mathbf{N}_j^{\cdot} = \begin{Vmatrix} 0 & E_J \\ R^*F_n & R^*F_{n^{\cdot}} \end{Vmatrix} N_j, \quad \mathbf{N}_j(0) = \mathbf{e}_j \tag{3.13}$$

$$(j = 1, \ldots, 2J)$$

$$\mu_j^{\cdot} = \frac{V^{\cdot}}{V}\mu_j - \frac{V}{2W} g_v \mathbf{N}_j, \quad \mu_j(0) = 0 \tag{3.14}$$

$$\mathbf{v}^{\cdot(\pi_k)} = \begin{Vmatrix} 0 & E_J \\ R^*F_n & R^*F_{n^{\cdot}} \end{Vmatrix} \mathbf{v}^{(\pi_k)} + \begin{Vmatrix} 0 \\ R^*F_{\pi_k} \end{Vmatrix}, \quad \mathbf{v}^{(\pi_k)}(0) = 0 \tag{3.15}$$

$$(k = 1, \ldots, K)$$

$$\mu^{\cdot(\pi_k)} = \frac{V^{\cdot}}{V}\mu^{(\pi_k)} - \frac{V}{2W}[g_v \mathbf{v}^{(\pi_k)} + g_{p_k}], \quad \mu^{(\pi_k)}(0) = 0 \tag{3.16}$$

In these equations we have used the notation $E_{2J} = \{\mathbf{e}_1, \ldots, \mathbf{e}_{2J}\}$, $N = \{\mathbf{N}_1, \ldots, \mathbf{N}_{2J}\}$, $\mu = \{\mu_1, \ldots, \mu_{2J}\}$, while the initial conditions $\mathbf{Q}(0)$, $\mathbf{Q}^{\cdot}(0)$ are of course considered to be known. The order of the system is $2(J+1) + 2J(2J+1) + (2J+1)K$.

Integration of this system of equations yields $N(T)$, $\mathbf{v}^{(\pi_k)}(T)$, $\mu(T)$, $\mu^{(\pi_k)}(T)$, and formulae (3.7), (3.11) and (3.10) may then be used to compute $\mathbf{n}^{(k)}(0)$, $\mathbf{n}^{\cdot(k)}(0)$, $m^{\cdot(k)}(0)$ and $\tau^{(k)}$. Formulae (3.3) now yield $\mathbf{n}(0)$, $\mathbf{n}^{\cdot}(0)$, $m(0)$, $m^{\cdot}(0)$ and τ, and these in turn, through formulae (2.6) and (2.7), give $\xi(0)$, $\xi^{\cdot}(0)$.

Finally, the initial values and period of the new periodic solution for parameter values $\mathbf{p} = \mathbf{P} + \pi$ are found from the formulae

$$\mathbf{q}(0) = \mathbf{Q}(0) + \xi(0), \quad \mathbf{q}^{\cdot}(0) = \varkappa \mathbf{q}^{\cdot *}(0), \quad T^* = T(\mathbf{p}) = T + \tau$$

$$\mathbf{q}^{\cdot *}(0) = \mathbf{Q}^{\cdot}(0) + \xi^{\cdot}(0), \quad \varkappa = \left[\frac{2W(\mathbf{q}(0), \mathbf{p})}{\mathbf{q}^{\cdot *T}(0) L(\mathbf{q}(0), \mathbf{p}) \mathbf{q}^{\cdot *}(0)}\right]^{1/2} \tag{3.17}$$

The correction factor $\varkappa$ has been introduced here to ensure the validity of the energy integral (1.2); if it were true that $\varkappa \equiv 1$ in (3.17), we would have $C \neq 0$ in (1.2), since the predictor gives only approximate values of the displacements in the initial conditions.

4. Corrector.

Orbits with initial conditions (3.17) are only approximately periodic, i.e., the differences $\mathbf{q}(T^*) - \mathbf{q}(0), \mathbf{q}^{\cdot}(T^*) - \mathbf{q}^{\cdot}(0)$ do not vanish, but they are small to second order with respect to the parameter increments π at the preceding step. The accuracy of the initial conditions and period is improved through the use of the corrector procedure.

Suppose now that (1.4) is a periodic solution of Eqs.(1.3), but such that in its neighbourhood in the phase space there is a periodic solution (1.6) corresponding to the same parameter values. Our aim is to find the latter, taking the former as an initial approximation.

Introducing local coordinates (displacements) in accordance with (2.1), we assume that $\pi \equiv 0$ and look for displacements in the initial conditions $\xi(0)$, $\xi^{\cdot}(0)$ and period τ.

The displacements $\xi(t)$ satisfy Eqs.(2.2) with integral (2.3). Introducing normal and tangential displacements $\mathbf{n}$ and m in accordance with formulae (2.6), we arrive at Eqs.(2.14) and (2.10).

We shall assume that the quantities $\Delta\mathbf{Q} = \mathbf{Q}(T) - \mathbf{Q}(0) \neq 0$, $\Delta\mathbf{Q}^{\cdot} = \mathbf{Q}^{\cdot}(T) - \mathbf{Q}^{\cdot}(0) \neq 0$ are small to the same order of magnitude as ξ, $\xi^{\cdot}$, τ. The boundary conditions for Eqs.(2.14), (2.10) are as follows:

$$\mathbf{n}(0) = \mathbf{n}(T) + R^*(0)\Delta\mathbf{Q} \tag{4.1}$$

$$\mathbf{n}^{\cdot}(0) = \mathbf{n}^{\cdot}(T) + R^{\cdot *}(0)\Delta\mathbf{Q} + R^*(0)\Delta\mathbf{Q}^{\cdot}$$

$$m(0) = m(T) + V(0)\tau + \mathbf{s}^*(0)\Delta\mathbf{Q} \tag{4.2}$$

$$m^{\cdot}(0) = m^{\cdot}(T) + V^{\cdot}(0)\tau + \mathbf{s}^{\cdot *}(0)\Delta\mathbf{Q} + \mathbf{s}^*(0)\Delta\mathbf{Q}^{\cdot}$$

where $R^{\cdot *}$ are the first J rows of the matrix dS^{-1}/dt and $s^{\cdot *}$ is the last row of the same matrix.

As in the predictor, the general solution of Eqs.(2.14) is expressed in the form (3.6) (of course, we must omit the superscripts k; $\mathbf{v}^{(\pi_k)} \equiv 0$). It then follows from the boundary conditions (4.1) that

$$\mathbf{v}(0) = \left\| \begin{matrix} \mathbf{n}(0) \\ \mathbf{n}^{\cdot}(0) \end{matrix} \right\| = -[N(T) - E_{2J}]^{-1} \left\| \begin{matrix} R^*(0)\Delta\mathbf{Q} \\ R^{\cdot *}(0)\Delta\mathbf{Q} + R^*(0)\Delta\mathbf{Q}^{\cdot} \end{matrix} \right\| \tag{4.3}$$

Now consider the boundary-value problem for the tangential displacements. Substitute the above normal displacements into (2.10). The general solution may then be written as (3.9), again omitting the indices k and putting $\mu^{(\pi_k)} \equiv 0$.

The initial displacement along the orbit may again be equated to zero:

$$m(0) = 0, \quad m^{\cdot}(0) = -(V(0)/(2W(0)))\, g_v(0)\mathbf{v}(0) \tag{4.4}$$

Then, from the first periodicity condition (4.2), we obtain the variation in the period:

$$\tau = -[\mu(T)\mathbf{v}(0) + \mathbf{s}^*(0)\Delta\mathbf{Q}]/V(0) \tag{4.5}$$

Direct differentiation now shows that the second boundary condition (4.2) is satisfied to the required degree of accuracy; the residual thus obtained may serve as a criterion for the accuracy of the corrector procedure.

Collecting all results of this section, we see that in order to determine the displacements in the initial conditions and period one must integrate a system of order $2(J+1)+2J(2J+1)$, namely equations (3.12)-(3.14), from $t=0$ to $t=T$. After integration one can use formulae (4.3)-(4.5) to calculate $\mathbf{n}(0)$, $\mathbf{n}^{\cdot}(0)$, $m(0)$, $m^{\cdot}(0)$, subsequently obtaining the displacements $\xi(0)$, $\xi^{\cdot}(0)$ and new initial conditions by means of formulae (2.6), (2.7) and (3.17).

The accuracy of the corrector procedure is determined by the relative error (where ε_1, ε_2 are weighting factors):

$$\varepsilon = [\varepsilon_1 \, |\xi(0)|/|\mathbf{q}(0)| + \varepsilon_2 \, |\xi^{\cdot}(0)|/|\mathbf{q}^{\cdot}(0)|] \tag{4.6}$$

If ε turns out to be smaller than the prescribed ε^*, the quantities (3.17) are accepted as the initial values and period of the required periodic solution. If the error (4.6) is still too great, one puts $\mathbf{Q}(0)=\mathbf{q}(0)$, $\mathbf{Q}^{\cdot}(0)=\mathbf{q}^{\cdot}(0)$, $T=T^*$ and repeats the corrector procedure until $\varepsilon<\varepsilon^*$. Note that the corrector as described guarantees accelerated quadratic convergence of the Newton type.

5. Retrograde periodic solutions. It was assumed previously that the velocity vector does not vanish at any point of the orbit. We shall now see how to relax that assumption and thereby generalize the method.

Let (1.4) to be a T-periodic solution (orbit) of Eqs.(1.3) with integral (1.2) and fixed parameter values $\mathbf{P}$. Suppose that it is not an equilibrium solution, i.e., in (1.3) we have $\mathbf{f}(\mathbf{Q},\mathbf{Q}^{\cdot},\mathbf{P})\not\equiv \mathbf{0}$ for $\mathbf{Q}^{\cdot}=\mathbf{0}$. Thus, at points of the orbit where the velocity $\mathbf{Q}^{\cdot}$ vanishes, the acceleration vector $\mathbf{Q}^{\cdot\cdot}$ does not vanish.

Let t_i be the times at which that happens (we may assume without loss of generality that $t_1\neq 0$), i.e.,

$$\mathbf{Q}^{\cdot}(t_i)=0,\quad \mathbf{Q}^{\cdot\cdot}(t_i)\neq 0,\quad i=1,\ldots,I,\quad 0<t_1<\ldots<t_I<T \tag{5.1}$$

This means that in the configuration space $\{\mathbf{Q}\}$ the point $\mathbf{Q}(t_i)$ lies on the (hyper)surface of zero velocity $W(\mathbf{Q},\mathbf{P})=0$ (see (1.2)), whereas the points $\mathbf{Q}(t)$, $0<|t-t_i|<\varepsilon$, do not.

In the neighbourhood of t_i the vector function $\mathbf{Q}^{\cdot}(t)$ and its absolute value $|\mathbf{Q}^{\cdot}(t)|$ admit of series expansions

$$\begin{aligned}\mathbf{Q}^{\cdot}(t) &= \mathbf{Q}^{\cdot\cdot}(t_i)(t-t_i)+\frac{1}{2}\mathbf{Q}^{\cdot\cdot\cdot}(t_i)(t-t_i)^2+\\ &\frac{1}{6}\mathbf{Q}^{\cdot\cdot\cdot\cdot}(t_i)(t-t_i)^3+\ldots \\ |\mathbf{Q}^{\cdot}(t)| &= \sqrt{\mathbf{Q}^{\cdot T}\mathbf{Q}^{\cdot}}=\sqrt{\mathbf{Q}^{\cdot\cdot T}(t_i)\mathbf{Q}^{\cdot\cdot}(t_i)}\,|t-t_i|+\ldots,\quad |t-t_i|<\varepsilon\end{aligned} \tag{5.2}$$

Thus we cannot speak of the existence of a limit $\lim \mathbf{s}(t)$ as $t\to t_i$, where $\mathbf{s}(t)=\mathbf{Q}^{\cdot}(t)/|\mathbf{Q}^{\cdot}(t)|$ is the unit vector tangent to the orbit. However, there are one-sided limits

$$\lim_{t\to t_i-0}\mathbf{s}(t)=-\lim_{t\to t_i+0}\mathbf{s}(t)=\frac{\mathbf{Q}^{\cdot\cdot}(t_i)}{|\mathbf{Q}^{\cdot\cdot}(t_i)|}\neq 0 \tag{5.3}$$

i.e., the tangent vector $\mathbf{s}(t)$ experiences a discontinuity of the first kind of $t=t_i$; in fact, it reverses direction when going through t_i. Consequently, $\mathbf{Q}(t_i)$ is a cusp of the orbit $\mathbf{Q}(t)$, and the normal (hyper)plane at that point coincides with the tangent plane to the surface $W(\mathbf{Q})=0$. In other words, the orbit "falls" and is "reflected" from the zero velocity surface in the direction of the normal to the latter /1/.

We shall show that this "normality of falling and reflection" property can be used to apply the corrector-predictor method in the case of retrograde solutions.

Let the orbit have a cusp $t=t_i$ at which conditions (5.1) are satisfied. Instead of (2.4) we define a new function:

$$V(t)=\delta(t)|\mathbf{Q}^{\cdot}(t)|,\quad |\mathbf{Q}^{\cdot}(t)|=\sqrt{\mathbf{Q}^{\cdot T}(t)\mathbf{Q}^{\cdot}(t)} \tag{5.4}$$

$$\delta(t)=\begin{cases}1, & 0\leqslant t<t_1\\ -1, & t_1\leqslant t<t_2\\ \ldots & \\ (-1)^i, & t_i\leqslant t<t_{i+1}\\ \ldots & \\ (-1)^I, & t_I\leqslant t\leqslant T\end{cases} \tag{5.5}$$

Clearly, the function (5.4) is continuous for all $t\in[0,T]$, vanishing only at $t=t_i$. Its expansion in the neighbourhood of the point t_i, i.e., for $t-t_i|<\varepsilon$ is

$$
\begin{aligned}
&V(t) = \delta(t)\,|t-t_i|\Big\{V^{\cdot}(t_i) + \frac{1}{2}\,V^{\cdot\cdot}(t_i)(t-t_i) + \\
&\quad \frac{1}{6}V^{\cdot\cdot\cdot}(t_i)(t-t_i)^2 + \dots\Big\} \\
&V^{\cdot}(t_i) = |\mathbf{Q}^{\cdot\cdot}(t_i)| = \sqrt{\mathbf{Q}^{\cdot\cdot T}(t_i)\,\mathbf{Q}^{\cdot\cdot}(t_i)} \neq 0 \\
&V^{\cdot\cdot}(t_i) = \frac{1}{V^{\cdot}}\,\mathbf{Q}^{\cdot\cdot\cdot T}\mathbf{Q}^{\cdot\cdot}\Big|_{t=t_i} \\
&V^{\cdot\cdot\cdot}(t_i) = \left[\frac{1}{V^{\cdot}}\,\mathbf{Q}^{\cdot\cdot\cdot\cdot T}\mathbf{Q}^{\cdot\cdot} + \frac{3}{4V^{\cdot}}\,\mathbf{Q}^{\cdot\cdot\cdot T}\mathbf{Q}^{\cdot\cdot\cdot} - \frac{3}{4V^{\cdot 3}}\,(\mathbf{Q}^{\cdot\cdot T}\mathbf{Q}^{\cdot\cdot\cdot})^2\right]\Big|_{t=t_i}
\end{aligned}
\tag{5.6}
$$

Consequently, $V(t)$ is analytic for all $t \in [0, T]$.

As before, we define the tangent vector $\mathbf{s}(t)$ by the formula $\mathbf{s}(t) = \mathbf{Q}^{\cdot}(t)/V(t)$. Then the vector function $\mathbf{s}(t)$ becomes analytic at $t \in [0, T]$, and at $t = t_i$ we have

$$
\begin{aligned}
&\mathbf{s}(t_i) = \frac{1}{V^{\cdot}}\,\mathbf{Q}^{\cdot\cdot}\Big|_{t=t_i}, \quad \mathbf{s}^{\cdot}(t_i) = \frac{1}{2}\left[\frac{1}{V^{\cdot}}\,\mathbf{Q}^{\cdot\cdot\cdot} - \frac{V^{\cdot\cdot}}{V^{\cdot 2}}\,\mathbf{Q}^{\cdot\cdot}\right]\Big|_{t=t_i} \\
&\mathbf{s}^{\cdot\cdot}(t_i) = \frac{1}{3}\left[\frac{1}{V^{\cdot}}\,\mathbf{Q}^{\cdot\cdot\cdot\cdot} - \frac{3}{2}\,\frac{V^{\cdot\cdot}}{V^{\cdot 2}}\,\mathbf{Q}^{\cdot\cdot\cdot} - \frac{V^{\cdot\cdot\cdot}}{V^{\cdot 2}}\,\mathbf{Q}^{\cdot\cdot} + \frac{3}{2}\,\frac{V^{\cdot\cdot}}{V^{\cdot 3}}\,\mathbf{Q}^{\cdot\cdot}\right]\Big|_{t=t_i}
\end{aligned}
\tag{5.7}
$$

It is clear that $\mathbf{s}(t)$ is simply the vector function defined in Sect.2 multiplied by $\delta(t)$.

As in Sect.2, we let $S(t)$ denote the non-singular transformation matrix to the new basis; its last column is $\mathbf{s}(t)$. In fact, $S(t)$ is simply the product of the analogous matrix in Sect.2 by $\delta(t)$. All previous properties of S are preserved, as is the form of the equations for the normal and tangential displacements. Note that $S(0) = (-1)^I S(T)$.

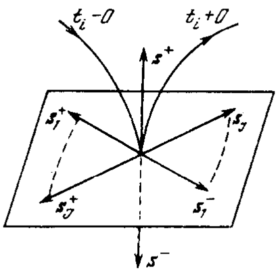

Fig.2

Fig.2 illustrates the orbit $\mathbf{Q}(t)$ in the configuration space $\{\mathbf{Q}\}$, and its normal (hyper)plane at the time $t = t_i$. The orbit does not cross its normal plane - the plane tangent to the surface $W = 0$ - but is "reflected" from it. It is obvious that at $t = t_i - 0$ and $t = t_i + 0$ the basis vector (indicated by superscripts minus and plus) reverse direction. However, this singularity has been removed by introducing the function $\delta(t)$.

The right-hand sides of Eqs.(2.14) for the normal displacements do not involve singularities.

Indeed, it follows from the form of the function F_n, $F_{n^{\cdot}}$ and F_{π} that there may be a singularity only in terms of the form

$$S^{-1}\,(V/(2W))\,(f_Q\mathbf{s} - 2\mathbf{s}^{\cdot}) = S^{-1}f_s \tag{5.8}$$

provided that at $t = t_i$ the function $V(t)$ has a first-order zero and $W(t)$ a second-order zero. The suspicion therefore arises that the term (5.8) involves a first-order pole. This is not the case, however. In fact, in deriving Eqs.(2.13) from (2.11) we used the relationship

$$\frac{1}{V}\,(f_Q\mathbf{s} - 2\mathbf{s}^{\cdot}) = -\frac{1}{V^{\cdot}}\,(f_Q\mathbf{s} + f_{Q^{\cdot}}\mathbf{s}^{\cdot} - \mathbf{s}^{\cdot\cdot}) + \frac{V^{\cdot\cdot}}{V}\,\mathbf{s}$$

where V is now defined by (5.4). Using the property $S^{-1}\mathbf{s} = \mathbf{e}$ (see (2.5)), we replace (5.8) by

$$\frac{V^2}{2W}\left[-\frac{1}{V^{\cdot}}\,S^{-1}(f_Q\mathbf{s} + f_{Q^{\cdot}}\mathbf{s}^{\cdot} - \mathbf{s}^{\cdot\cdot}) + \frac{V^{\cdot\cdot}}{V}\,\mathbf{e}\right]$$

or, finally,

$$R^{*}f_s = -\frac{1}{V^{\cdot}}\,\frac{1}{\mathbf{s}^T L\mathbf{s}}\,R^{*}\,(f_Q\mathbf{s} + f_{Q^{\cdot}}\mathbf{s}^{\cdot} - \mathbf{s}^{\cdot\cdot})$$

We have thus shown that in the case of retrograde periodic motions the right-hand sides of the equations for the normal displacements do not involve singularities.

In practice, however, there is no need to use formulae (5.7), (5.6) to calculate the vectors $\mathbf{s}$, $\mathbf{s}^{\cdot}$, $\mathbf{s}^{\cdot\cdot}$ at $t = t_i$. It suffices to evaluate them at a few points $t < t_i$ and $t > t_i$, and then use a suitable interpolation formula.

We will now consider the equation for the tangential displacement (2.10). To apply the corrector-predictor method we must find a particular solution with initial value $m(0) = 0$. However, even computer-aided numerical integration of this equation may prove quite difficult, since the coefficient of the highest-order derivative vanishes at $t = t_i$. It is therefore preferable to integrate the second-order equation obtained from (2.11):

$$m'' = s^* [(f_Q s + f_{Q'} s' - s'') m + (f_{Q'} s - 2s') m' + (f_Q R + f_{Q'} R' - R'') n + (f_{Q'} R - 2R') n' + f_P \pi]$$
$$m(0) = 0, \quad m'(0) = -(V(0)/(2W(0))) [g_n(0) n(0) + g_{n'}(0) n'(0) + g_P(0) \pi]$$

which involves no singularities.

Finally, we will consider the changes in the boundary conditions in the case of retrograde periodic motions. It follows from the formulae $S(0) = (-1)^I S(T)$, $V(0) = (-1)^I V(T)$ that in the boundary conditions (3.1) and (3.2) for the predictor and (4.1) and (4.2) for the corrector the displacements $n(T)$, $n'(T)$, $m(T)$, $m'(T)$ appear with a factor $(-1)^I$. We must therefore introduce this factor $(-1)^I$ before the functions evaluated at $t = T$ in Eqs.(3.7), (3.10), (4.3) and (4.5).

Thus, the algorithm has been fully generalized to the case of retrograde motions.

6. Stability. We will now briefly discuss the question of the stability of a given T-periodic solution (1.4). To this end we introduce perturbations ξ via formulae (2.1), assuming the parameters to be fixed, i.e., $\pi \equiv 0$ in (2.1). Then the matrices $f_Q, f_{Q'}$ in the variational Eqs.(2.1) and the column vectors $g_Q, g_{Q'}$ in their integral (2.3) are T-periodic.

It is clear that a periodic motion is stable under perturbations ξ, since its period depends on the initial conditions. However, one can consider the question of the oribital stability of a periodic motion /3/, i.e., we introduce perturbations n normal to the orbit, disregarding the question of stability under tangential perturbations m.

Let $X(t)$ be a matrix of fundamental solutions of system (2.13) (in which, of course, $\pi \equiv 0$), normalized by the initial condition $X(0) = E_{2J+2}$ - the identiy matrix of order $2J + 2$. We evaluate this matrix at $t = T$ and set up its characteristic equation

$$I_X(\rho) = \det \| X(T) - \rho E_{2J+2} \| = 0 \tag{6.1}$$

Since the original system is Lagrangian, the characteristic Eq.(6.1) is retrograde /3, 11/, i.e., the characteristic polynomial can be written

$$I_X(\rho) = \prod_{j=1}^{J+1} (\rho^2 - 2A_j\rho + 1) \tag{6.2}$$

Consider the structure of this polynomial. Since system (2.13) is equivalent to system (2.14), (2.15), we can verify that $I_X(\rho) = I_n(\rho) I_m(\rho)$, where $I_n(\rho)$ is the characteristic equation of system (2.14), which involves only the normal perturbations n, n'; $I_m(\rho)$ is the characteristic polynomial of the second-order differential equation $m'' = (V''(t)/V(t))m$. Direct integration of this equation shows that $I_m(\rho) = \rho^2 - 2\rho + 1$, i.e., as might be expected /1, 3/, the characteristic polynomial (6.1) has a pair of unit multipliers $\rho_{2J+1} = \rho_{2J+2} = 1$. But then the characteristic equation of the normal perturbations may be written in the form

$$I_n(\rho) = \det \| N(T) - \rho E_{2J} \| = \prod_{j=1}^{J} (\rho^2 - 2A_j\rho + 1) \tag{6.3}$$

where $N(t)$ is a matrix of fundamental solutions of the variational system of normal perturbations (2.14), normalized by the initial condition $N(0) = E_{2J}$.

Thus, we have proved the Lyapunov-Poincaré Theorem /3, 10, 11/ for the normal perturbations, according to which the characteristic polynomial is retrograde. As a corollary we obtain a criterion for orbital stability to a first approximation: the motion will be orbitally stable if and only if all roots of the equation $I_n(\rho) = 0$ (called multipliers) lie on the unit circle in the complex plane and the matrix $N(T)$ is reduced to diagonal form.

Note that the corrector procedure automatically calculates the matrix $N(T)$ at the last iteration. One advantage of this method is therefore that no additional calculations are required to analyse the stability of the motion to a first approximation, other than the construction of the motion itself.

We append a few remarks as to the limits of the applicability of our predictor-corrector method.

1°. If the matrix $\| N(T) - E_{2J} \|$ is singular, the initial conditions for the normal displacements cannot be determined through formulae (3.7) and (4.3). The equality $\det \| N(T) - E_{2J} \| = 0$ means that the matrix $N(T)$ has an eigenvalue equal to one, and since the characteristic Eq.(6.3) is retrograde the root $\rho_j = 1$ must have even multiplicity. A periodic solution of this type is naturally called critical /2, 3/. If a periodic solution is critical, the sufficient conditions of Poincaré's theorem for the existence of periodic solutions are violated. However, a solution being critical does not mean that it cannot be continued in any way; all it means is that the method described above is no longer applicable. A suitable predictor procedure may undoubtedly be devised by allowing for terms that are non-linear functions of the parameter displacements π; consideration of this case, however, is beyond the scope of this paper.

2°. Computer calculations will practically never achieve exact equality $\det \| N(T) - E_{2J} \| = 0$; hence, if the finite increments to π are suitably chosen, the computational scheme will not produce a singularity due to "overstepping" the critical orbit. Thus, using the same computational scheme one can obtain a new family generated by a critical orbit. In so doing, however, one can strictly speaking construct only one out of all possible families, i.e., one does not obtain a solution to the branching problem. Nevertheless, a combination of a suitable computational experiment of this sort with an analysis of the physical nature of the system may provide additional information.

In conclusion, we mention that the method proposed here has been used to investigate periodic motions arising from small librations perpendicular to the plane of revolution of the main attractive bodies in the three-dimenional circular restricted three-body problem*. (Karimov S.R. and Sokol'skii A.G., Periodic motions in the three-body problem, arising from spatial librations in the neighbourhood of Lagrangian solutions. Moscow, MAI, 1987. Deposited at VINITI, 24.08.87, 6182-B87.) As an indication of the efficiency of the method we note that the computation of a new motion took approximately 90 sec on the ES-1061 computer.

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